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SOME FIXED POINT THEOREMS FOR MULTI-VALUED MAPPINGS ON REFLEXIVE BANACH SPACES

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Abstract. Some fixed point theorems for multi- valued mappings satisfying an implicit relation which generalize the main results from [1] are proved.

1. Introduction

Let (X, d) be a metric space. We denote $CB(X)$ the set of all non-empty bounded closed subsets of (X, d) and by H the Hausdorff-Pompeiu metric on $CB(X)$

$$H(A, B) = \max\left\{\sup_{x\in A} d(x, B), \sup_{x\in B} d(x, A)\right\}$$

where A, $B \in CB(X)$ and $d(x, A) = \inf\{d(x, y) : y \in A\}$.

Let $T : (X, d) \to (X, d)$ be a multi-valued mapping. Denote $\Phi(T) = \{x \in X;\ x \in Tx\}$.

The purpose of this paper is to prove some fixed point theorems for multi-valued mappings satisfying an implicit relation which generalize the main results from [1].

2. Implicit relations

Let $\overline{\mathcal{F}}_6$ be the set of all real continuous functions $F(t_1, \ldots, t_6) : R_+^6 \to R$ satisfying the following conditions:

$\overline{F_1}$: F is non-decreasing in variable t_1 and non-increasing in variables t_5 and t_6;

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$\overline{F_2}$: there exists $h \in (0,1)$ such that for every $u \geq 0$, $v \geq 0$ with

$(F_a): \; F(u,v,v,u,u+v,0) \leq 0$ or $F_g : F(u,v,u,v,0,u+v) \leq 0$

we have $u \leq hv$

Ex.1[2]. $$F(t_1,\dots,t_6) = t_1 - k\max\left\{t_2,t_3,t_4,\frac{1}{2}(t_5+t_6)\right\}$$

where $k \in (0,1)$.

Ex.2[2]. $F(t_1,\dots,t_6) = t_2^2 - c_1\max\{t_2^2,t_3^2,t_4^2\} - c_2\max\{t_3t_5,t_4t_6\} - c_3t_5t_6$,

where $c_1 > 0; c_2, c_3 \geq 0$ and $c_1 + 2c_2 < 1$.

Ex.3[2]. $$F(t_1,\dots,t_6) = t_1^2 - t_1(at_2+bt_3+ct_4) - dt_5t_6$$

where $a > 0; b,c,d \geq 0$ and $a+b+c < 1$.

Ex.4[2]. $$F(t_1,\dots,t_6) = t_1^3 - at_1^2t_2 - bt_1t_3t_4 - ct_5^2t_6 - dt_5t_6^2$$

where $a > 0; b,c,d \geq 0$ and $a+b < 1$.

Ex.5[2]. $$F(t_1,\dots,t_6) = t_1^2 - at_2^2 - b\frac{t_5t_6}{t_3^2+t_4^2+1}$$

where $0 < a < 1$ and $b \geq 0$.

3. Main results

Definition 1. *A set valued-mapping $T : X \to CB(X)$ is said to have property (B) if the contractive condition defined on T enables us to construct a sequence $\{x_n\}$ for which $d(x_n, Tx_n) \to 0$ as $n \to \infty$.*

Theorem 1. *Let X be a reflexive Banach space and T a multi-valued mapping of X into the family of non-empty weakly compact subsets of X such that*

(1) $$F\Big(H(Tx,Ty),d(x,y),d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\Big) \leq 0$$

for all $x,y \in X$ and $F \in \overline{\mathfrak{F}_6}$. Then T has property (B).

Proof. Let x_0 be an arbitrary but fixed element of X. Choose $x_1 \in Tx_0$ such that $d(x_o,x_1) = d(x_0,Tx_0)$. This is possible because Tx_0 is a non-empty weakly compact subset of a reflexive Banach space, hence it is proximal [3, pp. 76]. Inductively we choose $x_n \in Tx_{n-1}$ so that $d(x_{n-1},x_n) = d(x_{n-1},Tx_{n-1})$. Now

$$F\Big(H(Tx_n,Tx_{n+1}),d(x_n,x_{n+1}),d(x_n,Tx_n),d(x_{n+1},Tx_{n+1}),d(x_n,Tx_{n+1}),d(x_{n+1},Tx_n)\Big) \leq 0$$

which implies succesivelly

$$F\Big(d(x_{n+1},Tx_{n+1}),d(x_n,x_{n+1}),d(x_n,Tx_n),d(x_{n+1},Tx_{n+1}),$$

$$d(x_n,x_{n+1}),d(x_{n+1},Tx_{n+1}),0\Big)\leq 0$$

$$F\Big(d(x_{n+1},Tx_{n+1}),d(x_n,Tx_n),d(x_n,Tx_n),d(x_n+1,Tx_n+1),$$

$$d(x_n,Tx_n)+d(x_{n+1},Tx_{n+1}),0\Big)\leq 0$$

By (F_a) we have $d(x_{n+1},Tx_{n+1})\leq hd(x_n,Tx_n)$ for each $n=0,1,2,\ldots$. From this is obvious that $d(x_{n+1},Tx_{n+1})\leq h^{n+1}d(x_0,T_0)$ and convergently $d(x_n,Tx_n)\to 0$ as $n\to\infty$.

Corollary 1. [1] *Let X be a reflexive Banach space and T a multi-valued mapping of X into the family of non-empty weakly compact subset of X such that*

$$(2)\quad H(Tx,Ty)\leq h\max\left\{d(x,y),d(x,Tx),d(y,Ty),\frac{1}{2}[d(x,Ty)+d(y,Tx)]\right\}$$

for all $x,y\in X$. Then T has property (B).

Proof. It follows from Theorem 1 and Ex. 1.

Theorem 2. *Let X be a reflexive Banach space and let T be a multi-valued mapping of X into the family of all non-empty weakly compact subests of X which satisfies (1). Then T has a fixed point in X.*

Proof. By Theorem 1 there exists a sequence $\{x_n\}\in X$ for which $d(x_{n+1},Tx_{n+1})\leq hd(x_n,Tx_n)$ for each n. Then by a routine calculation can show that $\{x_n\}$ is a Cauchy sequence in X.

Let $\{x_n\}$ converge to $p\in X$. Now, by (1), we have

$$F\Big(H(Tp,Tx_{n-1}),d(p,x_{n-1}),d(p,Tp),d(x_{n-1},Tx_{n-1}),d(p,Tx_{n-1}),$$

$$d(x_{n-1},Tp)\Big)\leq 0$$

$$F\Big(d(Tp,x_N),d(p,x_{n-1}),d(p,Tp),d(x_{n-1},x_n),d(p,x_n)+$$

$$+d(x_n,Tx_{n-1}),d(x_{n-1},T_p)\Big)\leq 0$$

Letting $n\to\infty$ we obtain by continuity of F that

$$F\Big(d(p,Tp),0,d(p,Tp),0,0,d(p,Tp)\Big)\leq 0$$

which implies by (F_b) that $d(p,Tp)=0$. Since Tp is closed this shows that $p\in Tp$.

Corollary 2. [1] *Let X be a reflexive Banach space and let T be a multi-valued mapping of X into the family of all non-empty weakly compact subsets of X which satisfies (2) for all $x, y \in X$. Then T has a fixed point in X.*

Proof. It follows form Theorem 2 and Ex. 1.

Theorem 3. *Let $T_1, T_2 : (X, d) \to CB(X)$ be two multi-valued mappings . If*

$$(3) \qquad F\Big(H(T_1x, T_2y), d(x, y), d(x, T_1x), d(y, T_2y), d(x, T_2y), d(y, T_1x)\Big) \leq 0$$

holds for all $x, y \in X$, where $F \in \overline{\mathcal{F}_6}$ and $\Phi(x_1) \neq \emptyset$ or $\Phi(T_2) \neq \emptyset$, then $\Phi(T_1) = \Phi(T_2)$.

Proof. If $u \in \Phi(T_1)$, then by (3) we have

$$F\Big(H(T_1u, T_2u), d(u, u), d(u, T_1u), d(u, T_2u), d(u, T_2u), d(u, T_1u)\Big) \leq 0.$$

By $d(u, T_2u) \leq H(T_1u, T_2u)$ and $\overline{(F_1)}$ it follows that

$$F\Big(d(u, T_2u), 0, 0, d(u, T_2u), d(u, T_2u), 0)\Big) \leq 0$$

which implies by (F_a) that $d(u, T_2) = 0$. Since T_2u is closed this shows that $u \in T_2u$ which implies $\Phi(T_1) \subset \Phi(T_2)$. analogous, $\Phi(T_2) \subset \Phi(T_1)$.

Theorem 4. *Let X be a reflexive Banach space and T_1 and T_2 be multi-valued mappings of X into the family of all non-empty weakly compact subsets of X. If inequality (3) holds for all $x, y \in X$, where $F \in \overline{\mathcal{F}_6}$, then (T_1) and (T_2) have a common fixed point and $\Phi(T_1) = \Phi(T_2)$.*

Proof. Let x_0 be an arbitrary but fixed element of X and chose $x_1 \in T_2x_0$ and $x_2 \in T_1x_1$ so that $d(x_0, x_1) = d(x_0, T_2x_0)$ and $d(x_1, x_2) = d(x_1, T_1x_1)$. Inductively, for $n \geq 1$ we choose $x_{2n+1} \in Tx_{2n}$ and $x_{2n+2} \in T_1x_{2n+1}$ so that $d(x_{2n}, x_{2n+1}) = d(x_{2n}, T_2x_{2n})$ and $d(x_{2n+1}, x_{2n+2}) = d(x_{2n+1}, T_1x_{2n+1})$. The existence of such points is guaranteed by the proximality of sets T_1x and T_2x. Now, by (3) we have

$$F\Big(H(T_1x_{2n+1}, T_2x_{2n+2}), d(x_{2n+1}, x_{2n+2}), d(x_{2n+1}, T_1x_{2n+1}), d(x_{2n+2}, T_2x_{2n+2}), d(x_{2n+1}, T_2x_{2n+2}), d(x_{2n+2}, T_1x_{2n+1})\Big) \leq 0$$

Hence we have successively

$$F\Big(d(x_{2n+2},T_2x_{2n+2}),d(x_{2n+1},x_{2n+2}),d(x_{2n+1},T_1x_{2n+1}),$$
$$d(x_{2n+2},T_2x_{2n+2}),d(x_{2n+1},x_{2n+2})+d(x_{2n+2},T_2x_{2n+2}),0)\le 0,$$
$$F\Big(d(x_{2n+2},T_2x_{2n+2}),d(x_{2n+1},x_{2n+1}),d(x_{2n+1},T_1x_{2n+1}),$$
$$d(x_{2n+2},T_2x_{2n+2}),d(x_{2n+1},T_1x_{2n+1})+d(x-2n+2,T_2x_{2n+2}),0\Big)\le 0.$$

which implies by (F_a) that

$$d(x_{2n+2},T_2x_{2n+2})\le hd(x_{2n+1},T_1x_{2n+1}).$$

Similarly, by (F_b), we have

$$d(x_{2n+1},T_1x_{2n+1})\le hd(x_{2n},T_2x_{2n}).$$

By a routine calculation we have that $\{x_n\}$ is a Cauchy sequence in X. Let $\{x_n\}$ converge to $p\in X$. By (3) and $\overline{(F_1)}$ we have successively

$$F\Big(H(T_1x_{2n+1},T_2p),d(x_{2n+1},p),d(x_{2n+1},T_1x_{2n+1}),$$
$$d(p,T_2p),d(x_{2n+1},T_2p),d(p,T_1x_{2n+1})\Big)\le 0$$
$$F\Big(d(x_{2n+2},T_2p),d(x_{2n+1},p),d(x_{2n+1},x_{2n+2}),$$
$$d(p,T_2p),d(x_{2n+1},T_2p),d(p,x_{2n+2})\Big)\le 0$$

Letting $n\to\infty$ we obtain by continuity of (F) that

$$F\Big(d(p,T_2p),0,0,d(p,T_2p),d(p,T_2p),0\Big)\le 0$$

which implies by (F_a) that $d(p,T_2p)=0$. Since T_2p is closed, $p\in T_2p$. Thus $p\in\Phi(T_2)$. By Theorem 3 $\Phi(T_1)=\Phi(T_2)$.

Corollary 3. [1] *Let X be a reflexive Banach space and S and T be the mappings of X into the family fo all non-empty weakly compact subsets of X. If for each $x,y\in X$ there exists $h\in(0,1)$ such that*

$$H(Sx,Ty)\le h\max\Big\{d(x,y),d(x,Sx),d(y,Ty),\frac{1}{2}\Big[d(x,Ty)+d(y,Tx)\Big]\Big\} \tag{4}$$

then there exists a common fixed point for S and T.

Proof. It follows from Theorem 4 and Ex. 1.

Theorem 5. *Let X be a reflexive Banach space and let $\{T_n\}$ be a sequence of multi-valued mappings such that $\{T_n\}$ maps X into the family of all* *non-empty wakly compact subsets of X. If for each $x, y \in X$ and for positive integer i*

$$F\Big(H(T_ix, T_{i+1}y), d(x,y), d(x,T_ix), d(y,T_{i+1}y), \qquad (5)$$

$$d(x, T_{i+1}y), d(y, T_ix)\Big) \leq 0$$

where $F\mathcal{F}_6$, then the sequence $\{T_n\}$ have a common fixed point and $\Phi(T_1) = \Phi(T_n)$ for $n = 2, 3, \dots$

Proof. It follows from Theorems 3 and 4.

Corollary 4. [1]. *Let X be a reflexive Banach space and let $\{T_n\}$ be a sequence of multi-valued mappings such that each $\{T_n\}$ maps X into the family of all non-empty weakly compact subsets of X. If for each $x, y \in X$ and for positive integers i and j*

$$H(T_ix, T_jy) \leq h \max\Big\{d(x,y), d(x,T_ix), d(y,T_jy), \qquad (6)$$

$$\frac{1}{2}\Big[d(x,T_jy) + d(y,T_ix)\Big]\Big\}$$

where $h \in (0,1)$, then there is a common fixed point for T_n.

4. References

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